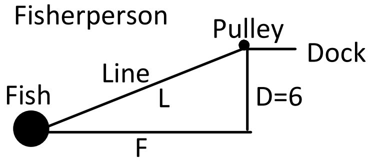
C12 - 4.1 - Pythag/Trig/Similar Δ Rel Rates Notes



$$\frac{dL}{dt} = 3\frac{m}{s}$$

$$\frac{dF}{dt}_{|F=8} = ?$$

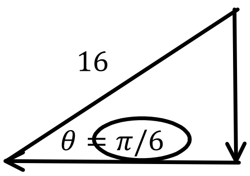
$$F^2 + D^2 = L^2$$

...

$$L = 10$$

$$F^2 + D^2 = L^2$$
$$F^2 + 6^2 = L^2$$
$$2F\frac{dF}{dt} + 0 = 2L\frac{dL}{dt}$$
$$2(8)\frac{dF}{dt} + 0 = 2(10)(-3)$$
$$\frac{dF}{dt} = -\frac{15}{2}\frac{m}{s}$$

The top of a 16 ft ladder slides down a wall at a rate of 3 ft/s. At what rate is the base of the ladder sliding away from the wall when the ladder is at a height of 8 ft on the wall and find theta.



$$\theta = \pi/6$$

$$y = 8^*$$
$$\frac{dy}{dt} = -3\frac{ft}{s}$$

$$x = 8\sqrt{3} \quad \frac{dx}{dt}|_{y=8} = ? = \sqrt{3}\frac{ft}{s}$$

$$x^2 + y^2 = c^2$$
$$x^2 + y^2 = 16^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$2(8\sqrt{3})\frac{dx}{dt} + 2(8)(-3) = 0$$
$$\frac{dx}{dt} = \frac{3}{\sqrt{3}} = \sqrt{3}\frac{ft}{s}$$

*We can substitute constants into the formula!

$$x^2 + y^2 = c^2$$
$$x^2 + 8^2 = 16^2$$
$$x = \sqrt{16^2 - 8^2}$$
$$x = \sqrt{192}$$
$$x = 8\sqrt{3}$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$
$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2\not{c}(0)$$

What is the rate the angle at the bottom of the ladder changing?

$$\sin\theta = \frac{8}{16}$$
$$\theta = \sin^{-1}(\frac{1}{2})$$
$$\theta = \frac{\pi}{6}$$

$$\cos\theta = \frac{x}{r}$$
$$\cos\theta = \frac{x}{16}$$
$$-\sin\theta\frac{d\theta}{dt} = \frac{1}{16}\frac{dx}{dt}$$
$$-\frac{8}{16}\frac{d\theta}{dt} = \frac{1}{16}\sqrt{3}$$
$$\frac{d\theta}{dt} = -\frac{\sqrt{3}}{8}\frac{rad}{s}$$

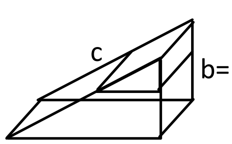
If possible put the constant on the bottom.

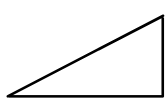
*Real life is in Radians.
Degrees are for children.
Unless you're taking Physics.

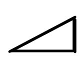
How fast is the area changing? $\frac{dA}{dt} = ?$

$$A = \frac{bh}{2}$$
$$\frac{dA}{dt} = \frac{1}{2}(\sqrt{3}(8) + (-3)(8\sqrt{3}))$$
$$\frac{dA}{dt} = -\frac{16\sqrt{3}}{2}$$
$$\frac{dA}{dt} = -8\sqrt{3}\frac{ft^2}{s}$$

Water-skier up a ramp.







$$b \approx 0$$

$$\frac{dc}{dt} = 2\frac{m}{s} \qquad \frac{db}{dt}_{|b\approx 0} = ?$$

a : b	Ratio stays
5 : 1	the same
0.005 : 0.001	

$$a^2 + b^2 = c^2$$
$$(5b)^2 + b^2 = c^2$$
$$26b^2 = c^2$$
$$52b\frac{db}{dt} = 2c\frac{dc}{dt}$$
$$52(1)\frac{db}{dt} = 2(\sqrt{26})(2)$$
$$\frac{db}{dt} = \frac{2\sqrt{26}}{13}\frac{m}{s}$$

$$\frac{b}{1} = \frac{a}{5}$$
$$a = 5b$$

$$a^2 + b^2 = c^2$$

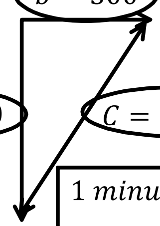
...

$$c = \sqrt{26}$$

C12 - 4.1 - Pythag/Area/Distance Rel Rates Notes

Train 'a' leaves Texas heading South at 10 m/s and train 'b' leaves heading East at 5 m/s 1 minute later? How far are they apart 1 minute after train 'b' departed? How fast are the trains are moving apart at that time?

$\frac{db}{dt} = 5$ $\quad b = 300$ $\quad \frac{dc}{dt}\Big|_{t=2} = ?$



$a = 1200$ $\quad C = 1236.9$

$\frac{da}{dt} = 10$

$1\ minutes = 60\ seconds$

$a = vt \quad d = vt + d_i \quad b = vt$

$a = 10 \times 60 + 10 \times 60 \quad b = 5 \times 60$

$a = 1200 \quad b = 300$

$$a^2+b^2 = c^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(1200)(10) + 2(300)(5) = 2(1136.9)\frac{dc}{dt}$$

$$27000 = 2473.9\frac{dc}{dt}$$

$$\frac{dc}{dt} = 10.9\frac{m}{s}$$

$$a^2 + b^2 = c^2$$

$$1200^2 + 300^2 = c^2$$

$$c = 300\sqrt{17}$$

$$c = 1236.9$$

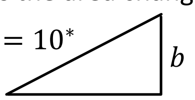
Exact Value

$$27000 = 600\sqrt{17}\frac{dc}{dt}$$

$$\frac{dc}{dt} = \frac{27000}{600\sqrt{17}}$$

$$\frac{dc}{dt} = \frac{45}{\sqrt{17}}\frac{m}{s}$$

How is the area changing?



$c = 10^*$, b, a

$\frac{db}{dt}\Big|_{a=6} = ?$

$\frac{da}{dt} = 2$

$$a^2+b^2 = c^2$$

$$a^2+b^2 = 10^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$2(6)(2) + 2(8)\frac{db}{dt} = 0$$

...

$$\frac{db}{dt} = -\frac{3}{2}\frac{cm}{s}$$

$$a^2 + b^2 = c^2$$

...

$$b = 8$$

$$A = \frac{1}{2}bh$$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt}h + \frac{dh}{dt}b\right)$$

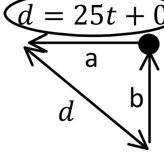
$$\frac{dA}{dt} = \frac{1}{2}\left((2)(6) + (-1.5)(8)\right)$$

$$\frac{dA}{dt} = 0\frac{cm^2}{s}$$

Distance Between.

$\frac{da}{dt} = 25\frac{km}{hr}$ $\quad \frac{dc}{dt}\Big|_{t=1pm} = ?$

$d = 25t + 0$ $\quad d_f = d_i + vt$



Noon (a=0,b=40)

a, b, d

$\frac{db}{dt} = -20\frac{km}{hr}$

$d = -20t + 20$

$$a^2+b^2 = c^2$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2(25)(25) + 2(20)(-20) = 2(32.0)\frac{dc}{dt}$$

$$\frac{dc}{dt} = 7.03\frac{km}{hr}$$

$$a^2 + b^2 = c^2$$

$$25^2 + 20^2 = c^2$$

...

$$c = 32.0$$

$y = mx + b$

$d = vt \quad d = d + vt$

$a = 25(1) \quad b = 40 - 20(1)$

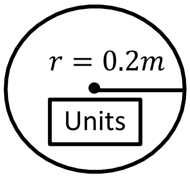
$a = 25 \quad b = 20$

C12 - 4.1 - Shapes Rel Rates Notes

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \qquad \frac{dA}{dr} = 2\pi r \times \frac{dr}{dr} \qquad \frac{dr}{dr} = 1$$

Find the rate of change.

If a pebble is dropped in a pond what is the rate at which the area of the circle is changing when the radius is 20cm if the radius is growing at a rate of 0.4 m/s.



$$\frac{dr}{dt} = 4 \qquad \left.\frac{dA}{dt}\right|_{r=20} = ?$$

Leibniz Notation

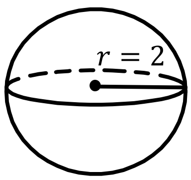
$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

Chain Rule

$$= 2\pi r \cdot (4)$$
$$= 2\pi(20)(4)$$
$$\frac{dA}{dt} = 160\pi \frac{m^2}{s}$$

Therefore the area is changing at a rate of $160\pi \frac{m^2}{s}$ when the radius is $20m$.

How fast is the radius of a sphere increasing when the radius is two meters if the volume of the balloon is decreasing at 256 meters cubed per second?



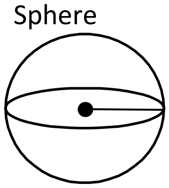
$$\frac{dV}{dt} = -256 \frac{m}{s^3} \qquad \left.\frac{dr}{dt}\right|_{r=2} = ?$$

$-ve$: Getting Smaller

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 3 \times \frac{4}{3}\pi r^{3-1} \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$-256 = 4\pi(2)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = -\frac{16}{\pi}\frac{m}{s}$$

Therefore the radius is changing at $-\frac{16}{\pi}\frac{m}{s}$ when the radius is 2 m.

Sphere

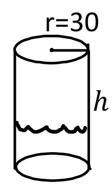


$$\frac{dV}{dt} = ? \qquad \left.\frac{dr}{dt}\right|_{SA=20} = 2$$

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi\left(\sqrt{\frac{100}{4\pi}}\right)^2 \frac{dr}{dt}$$
$$\frac{dV}{dt} = 4\pi \times \frac{100}{4\pi} \times 2$$
$$\frac{dV}{dt} = 200\frac{m^3}{s}$$

$$SA = 4\pi r^2$$
$$100 = 4\pi(2)^2$$
$$r = \sqrt{\frac{100}{4\pi}}$$
$$r = \frac{10}{2\sqrt{\pi}}\ m$$

Cylinder fill up.



$$V = \pi r^2 h$$
$$V = \pi(30)^2 h$$
$$V = 900\pi h$$
$$\frac{dV}{dt} = 900\pi \frac{dh}{dt}$$
$$\frac{dV}{dt} = 500\frac{cm^3}{s}$$
$$500 = 900\pi \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{5}{9\pi}\frac{cm}{s}$$

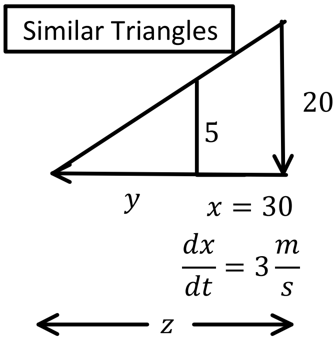
Find $\left.\frac{dA}{dC}\right|_C$

$$C = 2\pi r \qquad r = \frac{C}{2\pi}$$

$$A = \pi r^2$$
$$A = \pi\left(\frac{C}{2\pi}\right)^2$$
$$A = \frac{C^2}{4\pi}$$
$$\frac{dA}{dC} = \frac{1}{4\pi} 2C \frac{dC}{dt}$$

C12 - 4.1 - Similar Δ Rel Rates Notes

A 5 foot tall woman is walking away from a 20 foot lamp post at 3 m/s. What rate is her shadow increasing when she is 30 feet from the lamp post? How fast is the tip of her shadow moving?



$\frac{dx}{dt} = 3\frac{m}{s}$

$\left.\frac{dy}{dt}\right|_{x=30} = ?$

$\frac{5}{20} = \frac{y}{x+y}$

$5x + 5y = 20y$

$5x = 15y$

$x = 3y$

$\frac{dx}{dt} = 3\frac{dy}{dt}$

$3 = 3\frac{dy}{dt}$

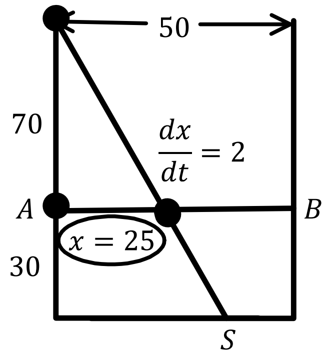
$\frac{dy}{dt} = 1\frac{ft}{s}$

$\frac{dz}{dt} = \frac{dy}{dt} + \frac{dx}{dt}$

$\frac{dz}{dt} = 1 + 3$

$\frac{dz}{dt} = 4\frac{m}{s}$

Tightrope-walker Shadow



$\frac{dx}{dt} = 2$

$\frac{dS}{dt}|_{x=25} = ?$

$\frac{x}{70} = \frac{S}{100}$

$x = \frac{7}{10}S$

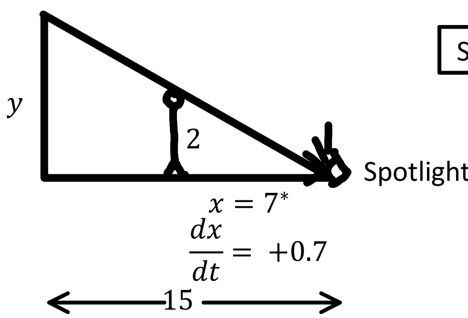
$\frac{dx}{dt} = \frac{7}{10}\frac{dS}{dt}$

$2 = \frac{7}{10}\frac{dS}{dt}$

$\frac{dS}{dt} = \frac{20}{7}\frac{m}{s}$

A 2 m tall person is walking away from a spotlight, 15 m from a wall, towards the wall at 0.7 m/s. How fast is the shadow on the wall changing when they are 7 m from the spotlight?

$\frac{dy}{dt}|_{x=7} = ?$



$x = 7^*$

$\frac{dx}{dt} = +0.7$

Similar Triangles

$\frac{y}{15} = \frac{2}{x}$

$xy = 30$

$\frac{dx}{dt}y + \frac{dy}{dt}x = 0$

$0.7(4.29) + \frac{dy}{dt}(7) = 0$

$\frac{dy}{dt} = -\frac{0.7(4.29)}{7}$

The shadow is decreasing at $\frac{dy}{dt} = -0.429\frac{m}{s}$

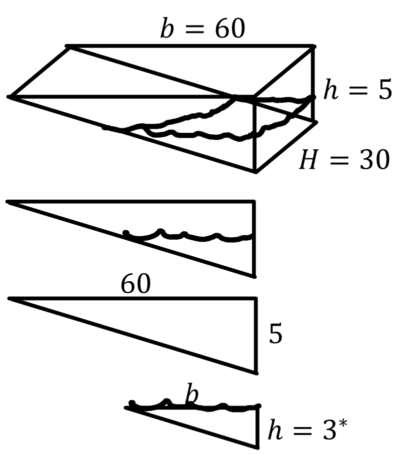
$\frac{y}{15} = \frac{2}{x}$

$\frac{y}{15} = \frac{2}{7}$

$y = 4.29$

*We can substitute constants into the formula!

Water Filling



$\frac{dh}{dt}_{|h=3} = ?$

$\frac{dV}{dt} = 5\frac{m^3}{min}$

$V = \frac{bh}{2}H$

$V = \frac{bh}{2}(30)$

$V = 15bh$

$V = 15(12h)h$

$V = 180h^2$

$\frac{DV}{dt} = 360h\frac{dh}{dt}$

$5 = 360(3)\frac{dh}{dt}$

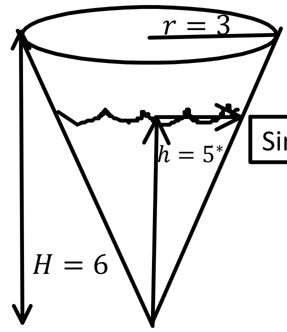
$\frac{dh}{dt} = \frac{1}{216}\frac{m}{min}$

$\frac{b}{60} = \frac{h}{5}$

$b = 12h$

C12 - 4.1 - Shapes/Similar Δ Cos Law Rel Rates Notes

A cone with radius 3 cm and height 6 cm is filling with water where the height of the water level is increasing at a rate of 0.2 cm/s. What is the rate the volume is increasing when the height of the water level is 5 cm.



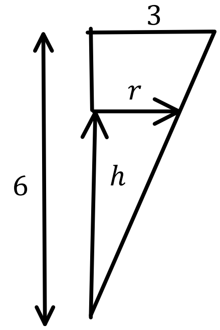
$\frac{dh}{dt} = 0.2$ $\quad \frac{dV}{dt}|_{h=5} = ?$

Similar Triangles

$\frac{H}{R} = \frac{h}{r}$

$\frac{6}{3} = \frac{h}{r}$

$r = \frac{h}{2}$



$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$

If it's draining into a cylinder use this #

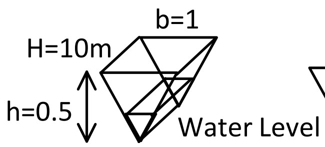
$V = \frac{1}{12}\pi h^3$

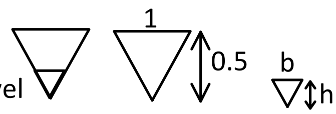
$\frac{dV}{dt} = 3 \times \frac{1}{12}\pi h^2 \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{1}{4}\pi(5)^2(0.2)$

$\frac{dV}{dt} = \frac{5\pi}{4}\frac{cm}{s}$

*We can't take this product so we must use similar triangles/other info $\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}h + \frac{dh}{dt}r^2\right)$





$\frac{dV}{dt} = 0.4\frac{m^3}{min}$ $\quad \frac{dh}{dt}_{|h=0.40} = ?$

$\frac{b}{1} = \frac{h}{0.5}$

$b = \frac{h}{\frac{1}{2}}$

$b = 2h$

$V = \frac{1}{2}bhH$

$V = \frac{1}{2}bh(10)$

$V = 5bh$

$V = 5(2h)h$

$V = 10h^2$

$\frac{dV}{dt} = 20h\frac{dh}{dt}$

$0.4 = 20(0.4)\frac{dh}{dt}$

$\frac{dh}{dt} = 0.05\frac{m}{min}$

A float plane rising at 30 degrees above the horizontal flies over a boat at an altitude of 100 m at 60 m/s. How fast is the distance between the boat and the plane increasing after five seconds?

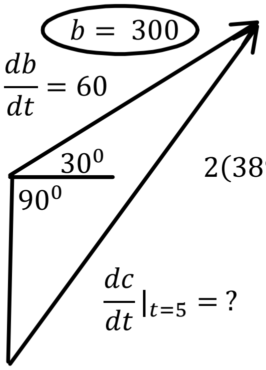
$d = vt$

$d = 60 \times 5$

$d = 300$

$b = 300$

$\frac{db}{dt} = 60$



$a = 100$

$\frac{dc}{dt}|_{t=5} = ?$

$c^2 = a^2 + b^2 - 2abcosC$

Cosine Law

$2c\frac{dc}{dt} = 0 + 2b\frac{db}{dt} - 2acosC\frac{db}{dt}$

$2(389.8)\frac{dc}{dt} = 0 + 2(300)(60) - 2(100)\left(-\frac{\sqrt{3}}{2}\right)(60)$

$\frac{dc}{dt} = 59.5\frac{m}{s}$

$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$120^0 = \frac{7\pi}{6}$

$c^2 = a^2 + b^2 - 2abcosC$

$c = \sqrt{100^2 + 300^2 - 2(100)(300)\cos\frac{7\pi}{6}}$

$c = 389.8\ m$

C12 - 4.1 - Demand Profit Max

$16\$$ *units sell* 20 *units.*
$14\$$ *units sell* 30 *units.*

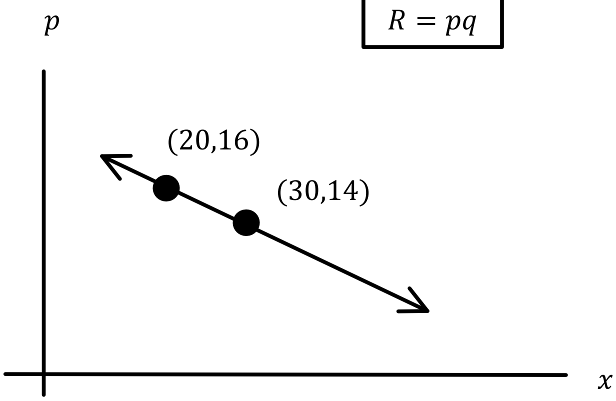
Find q to max R

$R = pq$

$C = 4x + 140$

$P = R - C$

$p = price$
$x = quantity$
$R = Revenue$
$C = Cost$
$P = Profit$



x	p	R	C	P
20	16	320	240	120
30	14	420	260	160

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{16 - 15}{20 - 30}$$
$$m = -\frac{1}{5}$$

Down $1 Sell 5 more

$$y - y_1 = m(x - x_1)$$
$$p - 16 = -\frac{1}{5}(x - 20)$$
$$p = -\frac{1}{5}x + 20$$

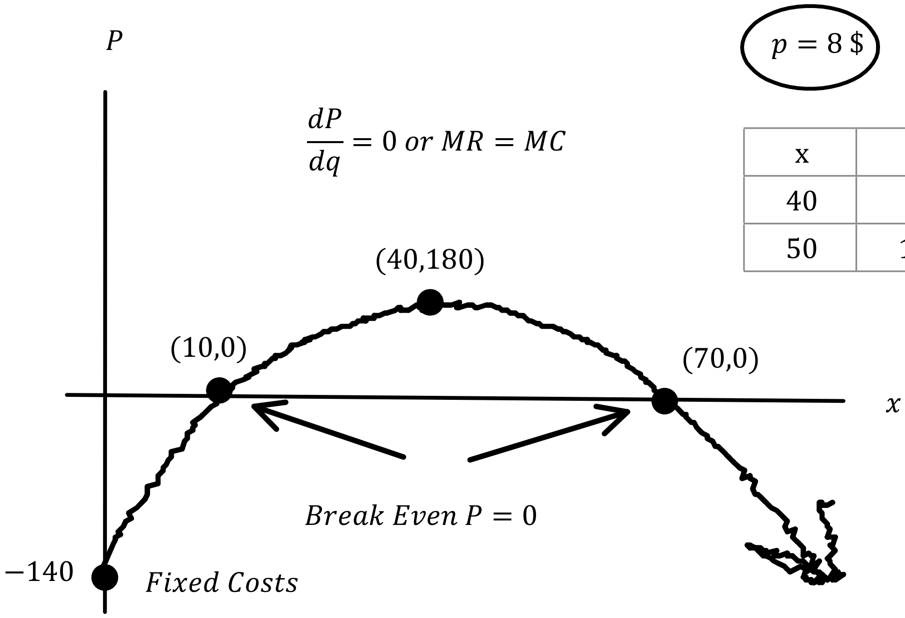
Demand Function

$$R = px$$
$$R = \left(-\frac{1}{5x} + 20\right)x$$
$$R = -\frac{1}{5}x^2 + 20x$$

$$p = -\frac{1}{5}x + 20$$
$$p = -\frac{1}{5} + 20$$
$$p = 8\ \$$$

$$P = R - C$$
$$P = -\frac{1}{5}x^2 + 20x - (4x + 140)$$
$$P = -\frac{1}{5}x^2 + 16x - 140$$
$$\frac{dP}{dx} = -\frac{2}{5}x + 16$$
$$0 = -\frac{2}{5}x + 16$$
$$x = 40\ units$$

$$\frac{dP}{dq} = 0 \text{ or } MR = MC$$



x	p	R	C	P
40	8	480	300	180
50	10	500	340	160

Max Profit

$$MR = MC$$
$$-\frac{2}{5}x + 20 = 4$$
$$x = 40$$

$$R = pq$$
$$R = (16 - 1x)(20 + 5x)$$
$$R = -5x^2 + 60x + 320$$
$$\frac{dR}{dx} = -10x + 60$$
$$0 = -10x + 60$$
$$x = 6$$

$x = \#\, p$ *decreases* $-\$1$ *q down,* $q + 5$

$$R = -5x^2 + 60x + 320$$
$$R = -5(6)^2 + 60(6) + 320$$
$$R = 500$$

6 price decrease, Max Revenue

Down $\$6$, *Rev* $= 500$

C12 - 4.1 - Growth Elasticity Max Rev Notes

$F'(500\$) = ?$ $\quad ; F(500\$)$ $\quad k = 5\%$ $\quad F = Pe^{kt}$

$$F = Pe^{kt}$$
$$F = Pe^{0.05t}$$
$$F' = Pe^{0.05t} \times 0.05$$

$$F = Pe^{kt}$$
$$F = Pe^{0.05t}$$
$$500 = Pe^{0.05t}$$
$$\frac{500}{P} = e^{0.05t}$$
$$\ln\left(\frac{500}{P}\right) = 0.05tlne$$
$$t = \frac{\ln\left(\frac{500}{P}\right)}{0.05}$$

$$F' = Pe^{0.05t} \times 0.05$$
$$F' = Pe^{0.05\frac{\ln\left(\frac{500}{P}\right)}{0.05}} \times 0.05$$
$$F' = Pe^{\ln\left(\frac{500}{P}\right)} \times 0.05$$
$$F' = P\left(\frac{500}{P}\right) \times 0.05$$
$$F' = 25\frac{\$}{year}$$

$$e^{\ln\left(\frac{500}{P}\right)} = \frac{500}{P}$$
$$e^{\ln_e a} = a$$

$q(p) = q$ $\quad$ *Quantity is a function of Price*

$$R = pq$$
$$\frac{DR}{dp} = \frac{dp}{dp}q + \frac{dq}{dp}p$$ Product

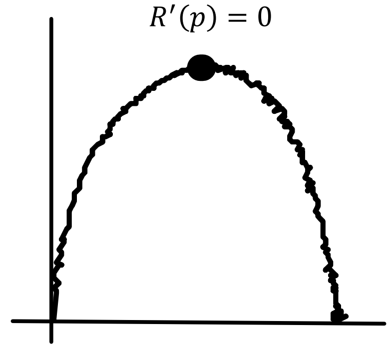
Rearrange $\quad \frac{dp}{dp} = 1$

$$\frac{DR}{dp} = p\frac{dq}{dp} + q$$

$$\frac{DR}{dp} = q\left(\frac{p}{q}\frac{dq}{dp} + 1\right)$$ Factor

$$\frac{DR}{dp} = q(E+1)$$

$$E(p) = \frac{p}{q}\frac{dq}{dp}$$ Elasticity



Price vs Quantity $\quad q + 50p^2 = 240$ $\quad$ Find q to max Rev

$$q + 5p^2 = 240$$
$$\frac{dq}{dp} + 10p\frac{dp}{dp} = 0$$
$$\frac{dp}{dp} = 1$$
$$\frac{dq}{dp} = -10p$$

Sell 10 less each increase in \$p

$$E(p) = \frac{p}{q}\frac{dq}{dp}$$
$$E(p) = \frac{p}{q} \times -10p$$
$$E(p) = -\frac{10p^2}{q}$$
$$-1 = -\frac{10p^2}{q}$$
$$q = 10p^2$$

$E(p) = -1$; @ max

$$q + 50p^2 = 240$$
$$10p^2 + 50p^2 = 240$$
$$60p^2 = 240$$
$$p^2 = 4$$
$$p = 2$$

$$q = 5p^2$$
$$q = 5(2)^2$$
$$q = 20$$